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LINEAR ALGEBRA AND ITS APPLICATIONS

Book review

Steve Kirkland, **Review of Nonnegative Matrices and Applications by R.B. Bapat and T.E.S. Raghavan**, *Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada, S4S 0A2*

The study of matrices with nonnegative entries began in the early part of this century with the pioneering work of Perron, who proved that a square matrix with positive entries has a positive dominant eigenvalue (the *Perron value*). Since that time, nonnegative matrix theory has evolved into a thriving enterprise. The subject has flourished in part because it has been informed by other areas of mathematics (for example combinatorics, numerical analysis, and probability) and because of its applications in other disciplines (such as economics, statistics, and operations research). These connections have enriched the subject both by furnishing interesting questions and new research directions, and by providing techniques with which to solve problems. Consequently, the literature involving ideas related to nonnegative matrices is remarkably diverse, and Bapat and Raghavan's *Nonnegative Matrices and Applications* presents a varied selection of results from this body of work. The book is intended both as a reference for the researcher in the area, and as a text for a first graduate course on the theory of nonnegative matrices.

Chapter 1 presents the basics of Perron–Frobenius theory for primitive, irreducible and reducible nonnegative matrices, and deals with some related topics such as nonsingular M-matrices and finite Markov chains. The authors approach the material from a variety of directions: Perron's theorem on positive matrices is established using Brouwer's fixed point theorem; the Perron–Frobenius theorem for irreducible matrices is proven using the theory of matrix games; directed graphs arise implicitly in the discussion of Markov chains.

In Chapter 2, the authors turn to the doubly stochastic matrices. One of the chapter's main themes is the Birkhoff–von Neumann theorem, which is proven in the first section, and is then used in obtaining a characterization of the nonnegative matrices having a doubly stochastic pattern; applications of the Birkhoff–von Neumann theorem arise in subsequent discussions of problems of

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optimal assignment, cooperative games and open shop scheduling. Permanents are also featured here, with proofs of the Frobenius–Konig theorem, a probabilistic algorithm for deciding whether a square (0, 1) matrix has a positive permanent, and a proof of the van der Warden conjecture which follows the approach of Egorychev.

Chapter 3 is mainly devoted to eigenvalue inequalities. It begins with simple bounds on the Perron value in terms of row sums, and gives some other Perron value inequalities which follow from the Information inequality. The inequalities of Levinger and Kingman are presented, as are results relating the circuit geometric means of a nonnegative matrix to its Perron value. A theorem of Cohen on the convexity of the Perron value of an irreducible matrix as a function of its main diagonal is also discussed. At this point, there is a change of pace, and the focus is placed on inequalities for positive semidefinite matrices. The authors present inequalities of Hadamard, Fiedler and Oppenheim, as well as majorization results for symmetric matrices, and the Series-Parallel inequality for positive semidefinite matrices.

The theme of symmetric matrices introduced in Chapter 3 is then continued in Chapter 4 on conditionally positive definite matrices (those symmetric matrices having a nonnegative quadratic form on the subspace orthogonal to the all ones vector). The properties of these matrices are discussed and applied to the problems of interpolating n distinct data points in the plane, and of understanding the behaviour of the entropy function of a multiparameter multinomial distribution. The conditionally positive definite matrices also arise naturally in the proof of a result characterizing the (entrywise) positive symmetric matrices having exactly one positive eigenvalue.

Chapter 5 surveys a variety of results in the theory of nonnegative matrices which involve combinatorial notions. These include the characterization of the connected graphs whose adjacency matrix has Perron value at most 2, the connection between those graphs and the graphs having a finite Coxeter group, spectral properties of matrices over the max algebra, and the Moore–Penrose inverse for matrices over the boolean algebra of two elements. This chapter also presents discussions of matroids, and of the mixed discriminant, a function which generalizes both the determinant and the permanent.

The problem of scaling the rows and columns of a nonnegative matrix in order to obtain desired row and column sums is central to Chapter 6. Such scaling problems are shown to arise in a number of contexts, including Markov chains, maximum likelihood estimation for contingency tables and economic models. The existence of such matrix scalings is proven using ideas from linear programming, while analytical tools are used to discuss the convergence of an iterative method for providing such a scaling.

Chapter 7 rounds out the book by discussing some mathematical models in economics, including those due to Leontief, Sraffa and von Neumann. The various models are outlined and the relationship with nonnegative matrices is highlighted, as are connections with P-matrices and N-matrices. The chapter concludes with some historical notes on the development of both the economic theory and the underlying mathematical ideas.

The book is clearly written, though there is a scattering of typographical errors, and the mathematics is presented rigorously (despite an error in the proof of Lemma 3.1.1). Most of the results are proven completely, although a few classical results, such as Brouwer's fixed point theorem and von Neumann's minimax theorem, are stated without proof; references are given for proofs and definitions which have been omitted. The authors have taken care in laying out the goals of the various sections, and in identifying the main results. Most of the sections conclude with remarks on the key results, and references to related work.

The selection of topics very much reflects the authors' own interests, which are eclectic, judging from the book's content; indeed a number of the results are due to one or both of the authors. There is an emphasis on presenting recent work (for example 60 of the 330 or so references are to publications appearing from 1990 onwards). Researchers in the area may find this book useful: it collects a number of recent developments in one volume, and it emphasizes the variety of different directions in which the study of nonnegative matrices has evolved. In this sense, *Nonnegative Matrices and Applications* provides an interesting counterpoint to other volumes (such as [1], or [2]) which cover the standard material on nonnegative matrices.

This book could reasonably serve as the basis for a graduate course covering topics from the theory of nonnegative matrices. The preface states that the only prerequisite is a first course in linear algebra and advanced calculus, but I suspect that a little more background in analysis might be useful. I certainly agree with the authors' proviso that familiarity with linear programming and statistics (particularly the former) is helpful in understanding some of the arguments. There are examples and applications throughout, and each chapter concludes with a selection of exercises (some 120 in total). These exercises range in difficulty from straightforward computational problems to proofs of results appearing in the recent literature, and hints are supplied for the more difficult problems. However, because of the focus on recent work, this book omits a number of older results from the theory of nonnegative matrices. For example, Chapter 3 contains few of the standard bounds on the Perron value (such as the improvements on Frobenius' row sum bounds due to Lederman, Ostrowski, and Brauer) and no discussion of the location of the non-Perron eigenvalues of a nonnegative matrix (despite the relevance of those eigenvalues to the convergence properties of Markov chains).

The authors have succeeded in conveying both their enthusiasm for the study of nonnegative matrices, and a sense of the richness and variety enjoyed by this vibrant subject. *Nonnegative Matrices and Applications* is a useful addition to the literature on nonnegative matrices, as it complements the standard works on the subject.

References

[1] A. Berman, R. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, 1994.
[2] E. Seneta, Non-negative Matrices and Markov Chains, 2nd ed., Springer, Berlin, 1981.